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Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Thin Hessenberg pairs

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ARTICLE INFO

Article history:

Received 23 November 2009

Accepted 12 January 2010

Available online 12 February 2010

Submitted by R.A. Brualdi

AMS classification:
15A04

Keywords:

Leonard pair

Hessenberg pair

Tridiagonal pair

q -Inverting pair

ABSTRACT

A square matrix is called *Hessenberg* whenever each entry below the subdiagonal is zero and each entry on the subdiagonal is nonzero. Let V denote a nonzero finite-dimensional vector space over a field \mathbb{K} . We consider an ordered pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ which satisfy both (i) and (ii) below.

- (i) There exists a basis for V with respect to which the matrix representing A is Hessenberg and the matrix representing A^* is diagonal.
- (ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is Hessenberg.

We call such a pair a *thin Hessenberg pair* (or *TH pair*). This is a special case of a *Hessenberg pair* which was introduced by the author in an earlier paper. We investigate several bases for V with respect to which the matrices representing A and A^* are attractive. We display these matrices along with the transition matrices relating the bases. We introduce an "oriented" version of A , A^* called a TH system. We classify the TH systems up to isomorphism.

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1. Introduction

This paper is about a linear algebraic object called a *Hessenberg pair*, which is a generalization of a *tridiagonal pair* [2,3]. We introduced Hessenberg pairs in [1]. In the present paper, we continue our investigation of Hessenberg pairs by focusing on a special case said to be *thin*. To define this case, we will use the following term. A square matrix is called *Hessenberg* whenever each entry below the

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subdiagonal is zero and each entry on the subdiagonal is nonzero. Throughout the paper, \mathbb{K} will denote a field.

Definition 1.1. Let V denote a nonzero finite-dimensional vector space over \mathbb{K} . By a *thin Hessenberg pair* (or *TH pair*) on V , we mean an ordered pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ which satisfy both (i) and (ii) below.

- (i) There exists a basis for V with respect to which the matrix representing A is Hessenberg and the matrix representing A^* is diagonal.
- (ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is Hessenberg.

Note 1.2. It is a common notational convention to use A^* to represent the conjugate-transpose of A . We are not using this convention. In a TH pair A, A^* the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

We now briefly summarize the paper. Let A, A^* denote a TH pair on V . We investigate six bases for V with respect to which the matrices representing A and A^* are attractive. We display these matrices along with the transition matrices relating the bases. We introduce an “oriented” version of A, A^* called a *TH system*. We classify the TH systems up to isomorphism. We will give a more detailed summary at the end of Section 2, after establishing some notations and reviewing some basic concepts.

2. TH systems

In our study of a TH pair, it is often helpful to consider a closely related object called a TH system. Before defining this notion, we make some definitions and observations. For the rest of the paper, fix an integer $d \geq 0$ and a vector space V over \mathbb{K} with dimension $d + 1$. Let $\text{End}(V)$ denote the \mathbb{K} -algebra of all linear transformations from V to V . Let $\text{Mat}_{d+1}(\mathbb{K})$ denote the \mathbb{K} -algebra consisting of all $(d + 1) \times (d + 1)$ matrices which have entries in \mathbb{K} . We index the rows and columns by $0, 1, \dots, d$. Suppose that $\{v_i\}_{i=0}^d$ is a basis for V . For $X \in \text{Mat}_{d+1}(\mathbb{K})$ and $Y \in \text{End}(V)$, we say X represents Y with respect to $\{v_i\}_{i=0}^d$ whenever $Yv_j = \sum_{i=0}^d X_{ij}v_i$ for $0 \leq j \leq d$. For $A \in \text{End}(V)$ and $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V | Av = \theta v\}$. In this case θ is called the *eigenvalue* of A corresponding to W . We say A is *diagonalizable* whenever V is spanned by the eigenspaces of A . We say A is *multiplicity-free* whenever A is diagonalizable and each eigenspace of A has dimension one.

Lemma 2.1. Let A, A^* denote a TH pair on V . Then each of A, A^* is multiplicity-free.

Proof. By Definition 1.1(ii), A is diagonalizable and so the minimal polynomial of A has no repeated roots. To show that each eigenspace of A has dimension one, we show that the minimal polynomial of A has degree $d + 1$. By Definition 1.1(i), there exists a basis for V with respect to which the matrix representing A is Hessenberg. Denote this matrix by B . On one hand, A and B have the same minimal polynomial. On the other hand, using the Hessenberg shape of B , we find I, B, B^2, \dots, B^d are linearly independent, so the minimal polynomial of B has degree $d + 1$. We conclude the minimal polynomial of A has degree $d + 1$ and it follows that A is multiplicity-free. The case of A^* is similar. \square

We recall a few more concepts from linear algebra. Let A denote a multiplicity-free element of $\text{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A . For $0 \leq i \leq d$, define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ ($0 \leq j \leq d$). Here I denotes the identity of $\text{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $I = \sum_{i=0}^d E_i$; (ii) $E_iE_j = \delta_{ij}E_i$ ($0 \leq i, j \leq d$); (iii) $V_i = E_iV$ ($0 \leq i \leq d$); (iv) $A = \sum_{i=0}^d \theta_i E_i$. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d). \quad (1)$$

Note that each of $\{A^i\}_{i=0}^d, \{E_i\}_{i=0}^d$ is a basis for the \mathbb{K} -subalgebra of $\text{End}(V)$ generated by A . Moreover $\prod_{i=0}^d (A - \theta_i I) = 0$.

We now define a TH system.

Definition 2.2. By a *thin Hessenberg system* (or *TH system*) on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d),$$

which satisfies (i)–(v) below.

- (i) Each of A, A^* is a multiplicity-free element of $\text{End}(V)$.
- (ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A .
- (iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^* .
- (iv) $E_i A^* E_j = \begin{cases} 0, & \text{if } i - j > 1 \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d).$
- (v) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } i - j > 1 \\ \neq 0, & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d).$

We call V the *underlying vector space* and say Φ is over \mathbb{K} .

We comment on how TH pairs and TH systems are related. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . For $0 \leq i \leq d$, let v_i (resp. v_i^*) denote a nonzero vector in $E_i V$ (resp. $E_i^* V$). Then the sequence $\{v_i\}_{i=0}^d$ (resp. $\{v_i^*\}_{i=0}^d$) is a basis for V which satisfies Definition 1.1(ii) (resp. Definition 1.1(i)). Therefore the pair A, A^* is a TH pair on V . Conversely, let A, A^* denote a TH pair on V . Then each of A, A^* is multiplicity-free by Lemma 2.1. Let $\{v_i\}_{i=0}^d$ (resp. $\{v_i^*\}_{i=0}^d$) denote a basis for V which satisfies Definition 1.1(ii) (resp. Definition 1.1(i)). For $0 \leq i \leq d$, the vector v_i (resp. v_i^*) is an eigenvector for A (resp. A^*); let E_i (resp. E_i^*) denote the corresponding primitive idempotent. Then $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a TH system on V .

Definition 2.3. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . Observe that A, A^* is a TH pair on V . We say this pair is *associated* with Φ .

Remark 2.4. With reference to Definition 2.3, conceivably a given TH pair is associated with many TH systems.

We now define the notion of *isomorphism* for TH systems.

Definition 2.5. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . Let W denote a vector space over \mathbb{K} with dimension $d + 1$, and let $\Psi := (B; \{F_i\}_{i=0}^d; B^*; \{F_i^*\}_{i=0}^d)$ denote a TH system on W . By an *isomorphism of TH systems* from Φ to Ψ we mean a vector space isomorphism $\gamma : V \rightarrow W$ such that

$$\gamma A = B \gamma, \quad \gamma A^* = B^* \gamma, \quad \gamma E_i = F_i \gamma \quad (0 \leq i \leq d), \quad \gamma E_i^* = F_i^* \gamma \quad (0 \leq i \leq d).$$

We say Φ and Ψ are *isomorphic* whenever there exists an isomorphism of TH systems from Φ to Ψ .

We now define the dual of a TH system.

Definition 2.6. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . Observe that $\Phi^* := (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ is a TH system on V . We call Φ^* the *dual* of Φ .

We recall some more terms. Let $\{v_i\}_{i=0}^d$ denote a basis for V . By the *inversion* of $\{v_i\}_{i=0}^d$ we mean the basis $\{v_{d-i}\}_{i=0}^d$. A square matrix is called *lower bidiagonal* whenever each nonzero entry lies on either the diagonal or the subdiagonal, and each entry on the subdiagonal is nonzero. A square matrix is called *upper bidiagonal* whenever its transpose is lower bidiagonal.

We now give a detailed summary of the paper. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . We investigate six bases for V that we find attractive. The first four are called the Φ -split basis, the Φ^* -split basis, the *inverted* Φ -split basis, and the *inverted* Φ^* -split basis. With respect to each of these bases, the matrix representing one of A , A^* is lower bidiagonal and the matrix representing the other is upper bidiagonal. The other two bases in our investigation are called the Φ -standard basis and the Φ^* -standard basis. A Φ -standard basis (resp. Φ^* -standard basis) satisfies Definition 1.1(i) (resp. Definition 1.1(ii)), subject to a certain normalization. For each of the six bases, we display the matrices representing A and A^* . We display some transition matrices relating these six bases. We associate with Φ a sequence of scalars called its *parameter array*. We show that Φ is determined up to isomorphism by its parameter array. Using this fact, we classify the TH systems up to isomorphism.

3. The eigenvalue sequences

Let Φ denote a TH system. In this section we associate with Φ two sequences of scalars called the *eigenvalue sequence* and the *dual eigenvalue sequence*. We describe some properties of these sequences that we will use later in the paper.

Definition 3.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . For $0 \leq i \leq d$, let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) corresponding to E_i (resp. E_i^*). We refer to $\{\theta_i\}_{i=0}^d$ as the *eigenvalue sequence* of Φ . We refer to $\{\theta_i^*\}_{i=0}^d$ as the *dual eigenvalue sequence* of Φ . We observe that $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in \mathbb{K} . Similarly $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and contained in \mathbb{K} .

Definition 3.2. Let A , A^* denote a TH pair. By an *eigenvalue sequence* of A , A^* , we mean the eigenvalue sequence of an associated TH system. By a *dual eigenvalue sequence* of A , A^* , we mean an eigenvalue sequence of the TH pair A^* , A . We emphasize that a given TH pair could have many eigenvalue and dual eigenvalue sequences.

Lemma 3.3. Let A , A^* denote a TH pair. Then the following (i) and (ii) hold.

- (i) Let $\{\theta_i\}_{i=0}^d$ and $\{\theta'_i\}_{i=0}^d$ denote eigenvalue sequences of A , A^* such that $\theta_0 = \theta'_0$. Then $\theta_i = \theta'_i$ for $0 \leq i \leq d$.
- (ii) Let $\{\theta_i^*\}_{i=0}^d$ and $\{\theta_i^{*'}\}_{i=0}^d$ denote dual eigenvalue sequences of A , A^* such that $\theta_0^* = \theta_0^{*'}$. Then $\theta_i^* = \theta_i^{*'}$ for $0 \leq i \leq d$.

Proof. (i) For $0 \leq i \leq d$, let V_i (resp. V'_i) denote the eigenspace of A corresponding to θ_i (resp. θ'_i). It suffices to show that $V_i = V'_i$ for $0 \leq i \leq d$. This follows once we show that $W_i = W'_i$ for $0 \leq i \leq d$, where $W_i = \sum_{h=0}^i V_h$ and $W'_i = \sum_{h=0}^i V'_h$. We prove this by induction on i . First assume that $i = 0$. Then $W_0 = V_0 = V'_0 = W'_0$ since $\theta_0 = \theta'_0$. Next assume that $1 \leq i \leq d$. By induction, we have $W_{i-1} = W'_{i-1}$. By Definition 1.1(ii) and since $\{\theta_h\}_{h=0}^d$ is an eigenvalue sequence of A , A^* , we find $A^*W_{i-1} \subseteq W_i$ and $A^*W_{i-1} \not\subseteq W_{i-1}$. Therefore $W_{i-1} + A^*W_{i-1} = W_i$. Similarly $W'_{i-1} + A^*W'_{i-1} = W'_i$. Comparing these equations using $W_{i-1} = W'_{i-1}$, we find $W_i = W'_i$. The result follows.

(ii) Apply (i) to Φ^* . \square

Lemma 3.4. Let A , A^* denote a TH pair on V . Fix a basis for V and let B (resp. B^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to that basis. Assume that B is Hessenberg and B^* is upper triangular. Then the sequence of diagonal entries $\{B_{ii}^*\}_{i=0}^d$ of B^* is a dual eigenvalue sequence of A , A^* .

Proof. We assume that A, A^* is a TH pair so A^* is multiplicity-free. We assume that B^* is upper triangular so the sequence $\{B_{ii}^*\}_{i=0}^d$ is an ordering of the eigenvalues of A^* . We show that this sequence is a dual eigenvalue sequence of A, A^* . For $0 \leq i \leq d$, let E_i^* denote the primitive idempotent of A^* corresponding to the eigenvalue B_{ii}^* . It suffices to show that Definition 2.2(v) holds. We make a few observations. Let $\{v_i\}_{i=0}^d$ denote the basis for V in the statement of the lemma. For $0 \leq i \leq d$, let W_i denote the subspace of V spanned by $\{v_h\}_{h=0}^i$. The matrix B^* is upper triangular so $A^*W_i \subseteq W_i$. The restriction of A^* to W_i has eigenvalues $\{B_{hh}^*\}_{h=0}^i$, so $W_i = E_0^*V + \cdots + E_i^*V$. Since B is Hessenberg, we have $AE_j^*V \subseteq W_{j+1}$ and $AE_j^*V \not\subseteq W_j$ for $0 \leq j \leq d-1$. We can now easily show that Definition 2.2(v) holds. Fix integers i, j ($0 \leq i, j \leq d$) such that $i-j \geq 1$. First assume that $i-j > 1$. From our observations above, $E_i^*AE_j^*V \subseteq E_i^*W_{j+1} = 0$. Therefore $E_i^*AE_j^*V = 0$ so $E_i^*AE_j^* = 0$. Next assume that $i-j = 1$. We show $E_i^*AE_j^* \neq 0$. By way of contradiction, assume that $E_i^*AE_j^* = 0$. By this and our earlier observations, we have $E_h^*AE_j^* = 0$ for $i \leq h \leq d$. Therefore $AE_j^* = \sum_{h=0}^d E_h^*AE_j^* = \sum_{h=0}^j E_h^*AE_j^*$, so $AE_j^*V \subseteq \sum_{h=0}^j E_h^*AE_j^*V \subseteq W_j$. This contradicts our above remarks so $E_i^*AE_j^* \neq 0$. The result follows. \square

4. The Φ -split basis

Let Φ denote a TH system on V . In this section we investigate a certain basis for V called the Φ -split basis. We will refer to the following notation.

Notation 4.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote the eigenvalue (resp. dual eigenvalue) sequence of Φ .

With reference to Notation 4.1, in our study of Φ we will use the following term. By a *decomposition* of V we mean a sequence $\{U_i\}_{i=0}^d$ of one-dimensional subspaces of V such that

$$V = U_0 + U_1 + \cdots + U_d \quad (\text{direct sum}).$$

For notational convenience, set $U_{-1} = 0$ and $U_{d+1} = 0$. We now describe a certain decomposition of V associated with Φ . For $0 \leq i \leq d$, define

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_0V + E_1V + \cdots + E_{d-i}V). \quad (2)$$

By [4, Lemma 2.4, Theorem 4.1], the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V . Moreover for $0 \leq i \leq d$, both

$$(A - \theta_{d-i}I)U_i = U_{i+1}, \quad (3)$$

$$(A^* - \theta_i^*I)U_i = U_{i-1}. \quad (4)$$

Setting $i = d$ in (2) we find $U_d = E_0V$. Combining this with (4) we find

$$U_i = (A^* - \theta_{i+1}^*I) \cdots (A^* - \theta_{d-1}^*I)(A^* - \theta_d^*I)E_0V \quad (0 \leq i \leq d). \quad (5)$$

Let η_0 denote a nonzero vector in E_0V . From (5) we find that for $0 \leq i \leq d$, the vector $(A^* - \theta_{i+1}^*I) \cdots (A^* - \theta_d^*I)\eta_0$ is a basis for U_i . By this and since $\{U_i\}_{i=0}^d$ is a decomposition of V , the sequence

$$(A^* - \theta_{i+1}^*I) \cdots (A^* - \theta_{d-1}^*I)(A^* - \theta_d^*I)\eta_0 \quad (0 \leq i \leq d)$$

is a basis for V .

Definition 4.2. With reference to Notation 4.1, a basis for V is said to be Φ -split whenever it is of the form

$$(A^* - \theta_{i+1}^*I) \cdots (A^* - \theta_{d-1}^*I)(A^* - \theta_d^*I)\eta_0 \quad (0 \leq i \leq d), \quad (6)$$

where $0 \neq \eta_0 \in E_0V$.

Lemma 4.3. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote a Φ -split basis for V , and let $\{w_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{w_i\}_{i=0}^d$ is a Φ -split basis for V .
- (ii) There exists $0 \neq c \in \mathbb{K}$ such that $w_i = cv_i$ for $0 \leq i \leq d$.

Proof. Routine. \square

With reference to Notation 4.1, our next goal is to describe the matrices representing A , A^* with respect to a Φ -split basis for V . We start with an observation. Let $1 \leq i \leq d$. By (4) we have $(A^* - \theta_i^* I)U_i = U_{i-1}$, and by (3) we have $(A - \theta_{d-i+1} I)U_{i-1} = U_i$. Therefore U_i is invariant under $(A - \theta_{d-i+1} I)(A^* - \theta_i^* I)$ and the corresponding eigenvalue is a nonzero element of \mathbb{K} . We denote this eigenvalue by ϕ_i . We call the sequence $\{\phi_i\}_{i=1}^d$ the split sequence of Φ . For notational convenience, set $\phi_0 = 0$ and $\phi_{d+1} = 0$.

Proposition 4.4. With reference to Notation 4.1, let B (resp. B^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to a Φ -split basis for V . Then

$$B = \begin{pmatrix} \theta_d & & & & \mathbf{0} \\ \phi_1 & \theta_{d-1} & & & \\ & \phi_2 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \phi_d & \theta_0 \end{pmatrix}, \quad B^* = \begin{pmatrix} \theta_0^* & 1 & & & \mathbf{0} \\ & \theta_1^* & 1 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & 1 \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}, \quad (7)$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ . In particular, B is lower bidiagonal and B^* is upper bidiagonal.

Proof. Follows from Definition 4.2 and the discussion prior to this proposition. \square

We give an alternate description of the Φ -split basis. To motivate this, we set $i = 0$ in (2) and find $U_0 = E_0^* V$. Combining this with (3) we find

$$U_i = (A - \theta_{d-i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I)E_0^* V \quad (0 \leq i \leq d). \quad (8)$$

Let η_0^* denote a nonzero vector in $E_0^* V$. From (8) we find that for $0 \leq i \leq d$, the vector $(A - \theta_{d-i+1} I) \cdots (A - \theta_d I)\eta_0^*$ is a basis for U_i . By this and since $\{U_i\}_{i=0}^d$ is a decomposition of V , the sequence

$$(A - \theta_{d-i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I)\eta_0^* \quad (0 \leq i \leq d)$$

is a basis for V . This basis is not a Φ -split basis in general, but we do have the following result.

Lemma 4.5. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{v_i\}_{i=0}^d$ is a Φ -split basis for V .
- (ii) There exists $0 \neq \eta_0^* \in E_0^* V$ such that

$$v_i = \frac{(A - \theta_{d-i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I)\eta_0^*}{\phi_1 \phi_2 \cdots \phi_i} \quad (0 \leq i \leq d), \quad (9)$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

Proof. Suppose that $\{v_i\}_{i=0}^d$ is a Φ -split basis for V . Then by Proposition 4.4, the matrices representing A and A^* with respect to $\{v_i\}_{i=0}^d$ are as shown in (7). From the matrix on the right in (7) we find $v_0 \in E_0^* V$. Taking $\eta_0^* = v_0$, we find (9) holds for $i = 0$. From the matrix on the left in (7), we find that (9) holds for $1 \leq i \leq d$. The result follows by Lemma 4.3. \square

We now give an alternate description of the split sequence of Φ .

Lemma 4.6. *With reference to Notation 4.1, let $\{\phi_i\}_{i=1}^d$ denote the split sequence of Φ . Then for $1 \leq i \leq d$, ϕ_i is the eigenvalue of $(A^* - \theta_i^* I)(A - \theta_{d-i+1} I)$ on the space U_{i-1} , where $\{U_i\}_{i=0}^d$ is from (2).*

Proof. Let $\{v_j\}_{j=0}^d$ denote a Φ -split basis for V . Recall that v_j spans U_j for $0 \leq j \leq d$, so it suffices to show that $(A^* - \theta_i^* I)(A - \theta_{d-i+1} I)v_{i-1} = \phi_i v_{i-1}$. This follows by Proposition 4.4 and a routine matrix computation. \square

We now give three characterizations of the Φ -split basis.

Proposition 4.7. *With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then this sequence is a Φ -split basis for V if and only if the following (i) and (ii) hold.*

- (i) $v_d \in E_0 V$.
- (ii) $A^* v_i = \theta_i^* v_i + v_{i-1}$ for $1 \leq i \leq d$.

Proof. Routine using Definition 4.2. \square

Proposition 4.8. *With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then this sequence is a Φ -split basis for V if and only if the following (i) and (ii) hold.*

- (i) $v_0 \in E_0^* V$.
- (ii) $Av_i = \theta_{d-i} v_i + \phi_{i+1} v_{i+1}$ for $0 \leq i \leq d-1$, where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

Proof. Routine using Lemma 4.5. \square

Proposition 4.9. *With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote a basis for V , and let C (resp. C^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to this basis. Then $\{v_i\}_{i=0}^d$ is a Φ -split basis for V if and only if the following (i)–(iii) hold.*

- (i) C is lower bidiagonal and C^* is upper bidiagonal.
- (ii) $C_{i-1,i}^* = 1$ for $1 \leq i \leq d$.
- (iii) $C_{dd} = \theta_0$ and $C_{00}^* = \theta_0^*$.

Proof. Suppose that $\{v_i\}_{i=0}^d$ is a Φ -split basis for V . By Proposition 4.4, conditions (i)–(iii) above hold. We have proved the proposition in one direction. For the other direction, suppose that conditions (i)–(iii) above hold. To show that $\{v_i\}_{i=0}^d$ is a Φ -split basis for V , we invoke Proposition 4.7. We show that Proposition 4.7(i), (ii) hold. Since C is lower bidiagonal with $C_{dd} = \theta_0$, v_d is an eigenvector of A corresponding to eigenvalue θ_0 . Therefore Proposition 4.7(i) holds. Since C is lower bidiagonal, C is Hessenberg. Since C^* is upper bidiagonal, C^* is upper triangular. Now apply Lemma 3.4 to conclude that $\{C_{ii}^*\}_{i=0}^d$ is a dual eigenvalue sequence of A , A^* . Therefore since $C_{00}^* = \theta_0^*$, by Lemma 3.3 we have $C_{ii}^* = \theta_i^*$ for $0 \leq i \leq d$. Since C^* is upper bidiagonal with $C_{i-1,i}^* = 1$ for $1 \leq i \leq d$, we have $A^* v_i = \theta_i^* v_i + v_{i-1}$ for $1 \leq i \leq d$. Therefore Proposition 4.7(ii) holds. The result follows. \square

5. Variations on the Φ -split basis

Let Φ denote a TH system on V . In the previous section we discussed the Φ -split basis for V . In this section we discuss three variations on this basis called the Φ^* -split basis, the inverted Φ -split basis, and the inverted Φ^* -split basis. The following lemma will be useful.

Lemma 5.1. With reference to Notation 4.1, let $\{\phi_i\}_{i=1}^d$ denote the split sequence of Φ . Then the split sequence of Φ^* is $\{\phi_{d-i+1}\}_{i=1}^d$.

Proof. Routine using the definition of the split sequence of Φ and Lemma 4.6. \square

We now discuss the Φ^* -split basis.

Lemma 5.2. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{v_i\}_{i=0}^d$ is a Φ^* -split basis for V .
- (ii) There exists $0 \neq \eta_0^* \in E_0^*V$ such that

$$v_i = (A - \theta_{i+1}I) \cdots (A - \theta_{d-1}I)(A - \theta_d I)\eta_0^* \quad (0 \leq i \leq d). \quad (10)$$

- (iii) There exists $0 \neq \eta_0 \in E_0V$ such that

$$v_i = \frac{(A^* - \theta_{d-i+1}^*I) \cdots (A^* - \theta_{d-1}^*I)(A^* - \theta_d^*I)\eta_0}{\phi_{d-i+1} \cdots \phi_{d-1}\phi_d} \quad (0 \leq i \leq d), \quad (11)$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

Proof. To prove (i) \Leftrightarrow (ii) apply Definition 4.2 to Φ^* . To prove (i) \Leftrightarrow (iii) apply Lemma 4.5 to Φ^* and use Lemma 5.1. \square

Proposition 5.3. With reference to Notation 4.1, let B (resp. B^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to a Φ^* -split basis for V . Then

$$B = \begin{pmatrix} \theta_0 & 1 & & & \mathbf{0} \\ & \theta_1 & 1 & & \\ & & \theta_2 & \ddots & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & 1 \\ & & & & \theta_d \end{pmatrix}, \quad B^* = \begin{pmatrix} \theta_d^* & & & & \mathbf{0} \\ \phi_d^* & \theta_{d-1}^* & & & \\ & \phi_{d-1} & \theta_{d-2}^* & & \\ & & \ddots & \ddots & \\ & & & \ddots & \phi_1 \\ \mathbf{0} & & & & \theta_0^* \end{pmatrix},$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ . In particular, B is upper bidiagonal and B^* is lower bidiagonal.

Proof. Apply Proposition 4.4 to Φ^* and use Lemma 5.1. \square

Lemma 5.4. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote a Φ -split basis for V , and let $\{w_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{w_i\}_{i=0}^d$ is a Φ^* -split basis for V .
- (ii) There exists $0 \neq c \in \mathbb{K}$ such that $w_i = \frac{c v_{d-i}}{\phi_{d-i+1} \cdots \phi_{d-1}\phi_d}$ for $0 \leq i \leq d$, where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

Proof. Compare (6) and (11). \square

We now turn our attention to the inverted Φ -split basis.

Lemma 5.5. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{v_i\}_{i=0}^d$ is an inverted Φ -split basis for V .
- (ii) There exists $0 \neq \eta_0 \in E_0V$ such that

$$v_i = (A^* - \theta_{d-i+1}^* I) \cdots (A^* - \theta_{d-1}^* I)(A^* - \theta_d^* I)\eta_0 \quad (0 \leq i \leq d).$$

(iii) There exists $0 \neq \eta_0^* \in E_0^* V$ such that

$$v_i = \frac{(A - \theta_{i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I)\eta_0^*}{\phi_1 \phi_2 \cdots \phi_{d-i}} \quad (0 \leq i \leq d),$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

Proof. Routine using Definition 4.2, Lemma 4.5, and the meaning of inversion. \square

Proposition 5.6. With reference to Notation 4.1, let B (resp. B^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to an inverted Φ -split basis for V . Then

$$B = \begin{pmatrix} \theta_0 & \phi_d & & & \mathbf{0} \\ & \theta_1 & \phi_{d-1} & & \\ & & \theta_2 & \ddots & \\ & & & \ddots & \phi_1 \\ \mathbf{0} & & & & \theta_d \end{pmatrix}, \quad B^* = \begin{pmatrix} \theta_d^* & & & & \mathbf{0} \\ 1 & \theta_{d-1}^* & & & \\ & 1 & \theta_{d-2}^* & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_0^* \end{pmatrix},$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ . In particular, B is upper bidiagonal and B^* is lower bidiagonal.

Proof. Routine using Proposition 4.4 and the meaning of inversion. \square

Lemma 5.7. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote a Φ -split basis for V , and let $\{w_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{w_i\}_{i=0}^d$ is an inverted Φ -split basis for V .
- (ii) There exists $0 \neq c \in \mathbb{K}$ such that $w_i = cv_{d-i}$ for $0 \leq i \leq d$.

Proof. Routine using Lemma 4.3 and the meaning of inversion. \square

We now turn our attention to the inverted Φ^* -split basis.

Lemma 5.8. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{v_i\}_{i=0}^d$ is an inverted Φ^* -split basis for V .
- (ii) There exists $0 \neq \eta_0^* \in E_0^* V$ such that

$$v_i = (A - \theta_{d-i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I)\eta_0^* \quad (0 \leq i \leq d).$$

(iii) There exists $0 \neq \eta_0 \in E_0 V$ such that

$$v_i = \frac{(A^* - \theta_{i+1}^* I) \cdots (A^* - \theta_{d-1}^* I)(A^* - \theta_d^* I)\eta_0}{\phi_{i+1} \cdots \phi_{d-1} \phi_d} \quad (0 \leq i \leq d),$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

Proof. Routine using Lemma 5.2 and the meaning of inversion. \square

Proposition 5.9. With reference to Notation 4.1, let B (resp. B^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to an inverted Φ^* -split basis for V . Then

$$B = \begin{pmatrix} \theta_d & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & \\ & 1 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_0 \end{pmatrix}, \quad B^* = \begin{pmatrix} \theta_0^* & \phi_1 & & & \mathbf{0} \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \phi_d \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix},$$

where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ . In particular, B is lower bidiagonal and B^* is upper bidiagonal.

Proof. Routine using Proposition 5.3 and the meaning of inversion. \square

Lemma 5.10. With reference to Notation 4.1, let $\{v_i\}_{i=0}^d$ denote a Φ -split basis for V , and let $\{w_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.

- (i) $\{w_i\}_{i=0}^d$ is an inverted Φ^* -split basis for V .
- (ii) There exists $0 \neq c \in \mathbb{K}$ such that $w_i = c\phi_1\phi_2 \cdots \phi_i v_i$ for $0 \leq i \leq d$, where $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

Proof. Routine using Lemma 5.4 and the meaning of inversion. \square

6. Classification of TH systems

In this section we classify the TH systems up to isomorphism.

Let Φ denote a TH system on V . In the previous sections we associated with Φ some sequences of scalars: the eigenvalue sequence, the dual eigenvalue sequence, and the split sequence. We now show that those sequences determine Φ up to isomorphism.

Lemma 6.1. Let Φ and Φ' denote TH systems over \mathbb{K} . Then the following are equivalent.

- (i) Φ and Φ' are isomorphic.
- (ii) Φ and Φ' share the same eigenvalue sequence, dual eigenvalue sequence, and split sequence.

Proof. (i) \rightarrow (ii). Clear.

(ii) \rightarrow (i). Without loss of generality, assume that Φ is the TH system from Definition 2.2. Write $\Phi' = (A'; \{E'_i\}_{i=0}^d; A^{*'}; \{E_i^{*'}\}_{i=0}^d)$. Let V' denote the vector space underlying Φ' . Let $\{v_i\}_{i=0}^d$ (resp. $\{v'_i\}_{i=0}^d$) denote a Φ -split (resp. Φ' -split) basis for V (resp. V'). Let γ denote the vector space isomorphism from V to V' which sends v_i to v'_i for $0 \leq i \leq d$. Since Φ and Φ' share the same eigenvalue, dual eigenvalue, and split sequences, the matrix representing A with respect to $\{v_i\}_{i=0}^d$ and the matrix representing A' with respect to $\{v'_i\}_{i=0}^d$ are identical. Thus $\gamma A = A' \gamma$. Similarly $\gamma A^* = A^{*'} \gamma$. Moreover, using (1) we find $\gamma E_i = E'_i \gamma$ and $\gamma E_i^* = E_i^{*'} \gamma$ for $0 \leq i \leq d$. Therefore γ is an isomorphism of TH systems from Φ to Φ' and so Φ and Φ' are isomorphic. \square

Lemma 6.1 motivates the following definition.

Definition 6.2. Let Φ denote a TH system on V . By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$, where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue (resp. dual eigenvalue) sequence of Φ and $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ .

In the following theorem, we classify the TH systems up to isomorphism.

Theorem 6.3. Let

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d) \tag{12}$$

- (i) $\theta_i \neq \theta_j$ if $i \neq j$ ($0 \leq i, j \leq d$).
- (ii) $\theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq d$).
- (iii) $\phi_i \neq 0$ ($1 \leq i \leq d$).

Proof. To prove the theorem in one direction, let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system over \mathbb{K} with parameter array (12). We show that conditions (i)–(iii) above hold. Conditions (i), (ii) hold by Definition 3.1 and condition (iii) holds by the observation prior to Proposition 4.4. We are done with the proof in one direction.

$$B = \begin{pmatrix} \theta_d & & & & \mathbf{0} \\ \phi_1 & \theta_{d-1} & & & \\ & \phi_2 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \phi_d \\ \mathbf{0} & & & & \theta_0 \end{pmatrix}, \quad B^* = \begin{pmatrix} \theta_0^* & 1 & & & \mathbf{0} \\ & \theta_1^* & 1 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & 1 \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}.$$
$$U_i + U_{i+1} + \cdots + U_d = E_0V + E_1V + \cdots + E_{d-i}V. \quad (13)$$
$$L = \{v \in V | Yv = 0\}. \quad (14)$$

Since $(A - \theta_{d-h}I)U_h = U_{h+1}$ for $0 \leq h \leq d$, we have $YU_j = 0$ for $i \leq j \leq d$, so $YK = 0$. Combining this with (14), we find $K \subseteq L$. We now have $K = L$ and the claim is proved. We now prove condition (iv). Fix integers i, j ($0 \leq i, j \leq d$) such that $i - j \geq 1$. First assume that $i - j > 1$. Observe that $(A^* - \theta_h^*)U_h = U_{h-1}$ for $0 \leq h \leq d$. Using this fact and (13), we find $A^*E_jV \subseteq A^* \sum_{h=0}^j E_hV = A^* \sum_{h=d-j}^d U_h \subseteq \sum_{h=d-j-1}^d U_h = \sum_{h=0}^{j+1} E_hV$. Therefore $E_iA^*E_jV \subseteq E_i \sum_{h=0}^{j+1} E_hV = 0$ so $E_iA^*E_j = 0$. Next assume that $i - j = 1$. We show that $E_iA^*E_j \neq 0$. By way of contradiction, assume that $E_iA^*E_j = 0$. Using these comments and (13), we find $A^*U_{d-j} \subseteq A^* \sum_{h=d-j}^d U_h = A^* \sum_{h=0}^j E_hV = (\sum_{k=0}^d E_k)A^*(\sum_{h=0}^j E_h)V \subseteq \sum_{h=0}^j E_hV = \sum_{h=d-j}^d U_h$. This contradicts our earlier remark that $(A^* - \theta_{d-j}^*)U_{d-j} = U_{d-j-1}$. Therefore $E_iA^*E_j \neq 0$ and Definition 2.2(iv) holds. The proof for Definition 2.2(v) is similar and omitted. We have now shown that Φ is a TH system on V . By Definition 3.1, Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By Proposition 4.9, $\{u_i\}_{i=0}^d$ is a Φ -split basis for V . Therefore Φ has split sequence $\{\phi_i\}_{i=1}^d$ by Proposition 4.4. The TH system Φ is unique up to isomorphism by Lemma 6.1. \square

We finish this section with a comment.

Lemma 6.4. Let Φ denote a TH system on V with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Then the dual TH system Φ^* has parameter array $(\{\theta_i^*\}_{i=0}^d, \{\theta_i\}_{i=0}^d, \{\phi_{d-i+1}\}_{i=1}^d)$.

Proof. Immediate from Definitions 2.6, 3.1 and Lemma 5.1. \square

7. The scalar ν

In this section we introduce a scalar ν that will help us describe TH systems. We start by updating Notation 4.1.

Notation 7.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ denote the parameter array of Φ .

Lemma 7.2. With reference to Notation 7.1, let \mathcal{D} (resp. \mathcal{D}^*) denote the \mathbb{K} -subalgebra of $\text{End}(V)$ generated by A (resp. A^*). Fix $0 \neq \eta_0 \in E_0V$ and $0 \neq \eta_0^* \in E_0^*V$. Then each of the maps

$$\begin{array}{ll} \mathcal{D} \rightarrow V & \mathcal{D}^* \rightarrow V \\ X \mapsto X\eta_0^* & X \mapsto X\eta_0 \end{array}$$

is an isomorphism of \mathbb{K} -vector spaces.

Proof. For the map on the left, use the fact that (10) is a basis for V . For the map on the right, use the fact that (6) is a basis for V . \square

Lemma 7.3. With reference to Notation 7.1, each of the maps

$$\begin{array}{ll} E_0^*V \rightarrow E_0V & E_0V \rightarrow E_0^*V \\ v \mapsto E_0v & v \mapsto E_0^*v \end{array}$$

is an isomorphism of \mathbb{K} -vector spaces.

Proof. Use Lemma 7.2 and the fact that each of E_0V, E_0^*V has dimension 1. \square

Consider the maps in Lemma 7.3. If we compose the map on the left with the map on the right, then the resulting map acts on E_0V as a nonzero scalar multiple of the identity. Denote the scalar by α . If we now compose the map on the right with the map on the left, then the resulting map acts on E_0^*V as α times the identity. We define ν to be the reciprocal of α . Observe that ν is nonzero.

Lemma 7.4. With reference to Notation 7.1, the following (i) and (ii) hold.

- (i) $\nu E_0 E_0^* E_0 = E_0$.
- (ii) $\nu E_0^* E_0^* E_0^* = E_0^*$.

Proof. Clear from the definition of ν . \square

We mention one significance of ν .

Lemma 7.5. With reference to Notation 7.1, $\text{tr}(E_0 E_0^*) = \nu^{-1}$.

Proof. Consider the equation in Lemma 7.4(i). Take the trace of each side and then simplify using the fact that $\text{tr}(E_0) = 1$ and $\text{tr}(E_0 E_0^* E_0) = \text{tr}(E_0 E_0^* E_0^*) = \text{tr}(E_0 E_0^*)$. The result follows. \square

We now express ν in terms of the parameter array of Φ .

Lemma 7.6. *With reference to Notation 7.1,*

$$\nu = \frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{\phi_1 \phi_2 \cdots \phi_d}.$$

Proof. Fix $0 \neq \eta_0 \in E_0 V$ and let $\{v_i\}_{i=0}^d$ denote the corresponding Φ -split basis for V from Definition 4.2. Observe that $v_d = \eta_0$. By Lemma 7.4, we have $\nu E_0 E_0^* E_0 = E_0$. Applying each side of this equation to v_d and using $v_d \in E_0 V$, we find $\nu E_0 E_0^* v_d = v_d$. By (1) we have $E_0 = \psi^{-1} \prod_{i=1}^d (A - \theta_i I)$, where $\psi = \prod_{i=1}^d (\theta_0 - \theta_i)$. Similarly we have $E_0^* = \psi^{*-1} \prod_{i=1}^d (A^* - \theta_i^* I)$, where $\psi^* = \prod_{i=1}^d (\theta_0^* - \theta_i^*)$. By Proposition 4.4, we find $\prod_{i=1}^d (A - \theta_i I) v_0 = \phi v_d$, where $\phi = \prod_{i=1}^d \phi_i$. Similarly $\prod_{i=1}^d (A^* - \theta_i^* I) v_d = v_0$. Combining these facts, we find $\nu \phi v_d = \psi \psi^* v_d$. Now $\nu \phi = \psi \psi^*$ since $v_d \neq 0$. The result follows. \square

8. The Φ -standard basis

Let Φ denote a TH system on V . In this section we investigate a certain basis for V called the Φ -standard basis.

Lemma 8.1. *With reference to Notation 7.1, let $0 \neq \eta_0 \in E_0 V$. Then the sequence*

$$E_0^* \eta_0, E_1^* \eta_0, \dots, E_d^* \eta_0 \quad (15)$$

is a basis for V .

Proof. This follows from Lemma 7.2 and the fact that $\{E_i^*\}_{i=0}^d$ is a basis for \mathcal{D}^* . \square

Definition 8.2. With reference to Notation 7.1, a basis for V is called Φ -standard whenever it is of the form (15), where $0 \neq \eta_0 \in E_0 V$.

Lemma 8.3. *With reference to Notation 7.1, let $\{v_i\}_{i=0}^d$ denote a Φ -standard basis for V , and let $\{w_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent.*

- (i) $\{w_i\}_{i=0}^d$ is a Φ -standard basis for V .
- (ii) There exists $0 \neq c \in \mathbb{K}$ such that $w_i = cv_i$ for $0 \leq i \leq d$.

Proof. Routine. \square

Definition 8.4. With reference to Notation 7.1, let H (resp. D^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to a Φ -standard basis for V .

Our next goal is to describe the matrices H and D^* . We start with D^* .

Lemma 8.5. *With reference to Notation 7.1 and Definition 8.4, the matrix D^* is diagonal with entries $D_{ii}^* = \theta_i^*$ for $0 \leq i \leq d$.*

Proof. Recall $A^* E_i^* = \theta_i^* E_i^*$ for $0 \leq i \leq d$. The result follows. \square

We now turn our attention to the matrix H . We recall some linear algebraic terms. Suppose we are given two bases for V , written $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$. By the *transition matrix* from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$, we mean the matrix T in $\text{Mat}_{d+1}(\mathbb{K})$ that satisfies $v_j = \sum_{i=0}^d T_{ij} u_i$ for $0 \leq j \leq d$. We recall a few properties

of transition matrices. Let T denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Then T^{-1} exists, and equals the transition matrix from $\{v_i\}_{i=0}^d$ to $\{u_i\}_{i=0}^d$. Let $\{w_i\}_{i=0}^d$ denote a basis for V , and let S denote the transition matrix from $\{v_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$. Then TS is the transition matrix from $\{u_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$. Let $A \in \text{End}(V)$, and let M (resp. N) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A with respect to $\{u_i\}_{i=0}^d$ (resp. $\{v_i\}_{i=0}^d$). Then $M = TNT^{-1}$.

Lemma 8.6. With reference to Notation 7.1 and Definition 8.4, let $\{u_i\}_{i=0}^d$ denote a Φ -split basis for V , and let $\{v_i\}_{i=0}^d$ denote a Φ -standard basis for V . Let B (resp. B^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to $\{u_i\}_{i=0}^d$. Let $T \in \text{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{v_i\}_{i=0}^d$ to $\{u_i\}_{i=0}^d$. Then the following (i) and (ii) hold.

- (i) $H = TBT^{-1}$.
- (ii) $D^* = TB^*T^{-1}$.

Proof. Follows from the comments prior to this lemma. \square

Recall that we are trying to find the matrix H . To do this we use the equations in Lemma 8.6. In these equations the matrices B and B^* were found in Proposition 4.4 and the matrix D^* was found in Lemma 8.5. We use Lemma 8.6(ii) to find the matrix T and then use Lemma 8.6(i) to find the matrix H .

Lemma 8.7. With reference to Notation 7.1, fix $0 \neq \eta_0 \in E_0V$. Let $\{u_i\}_{i=0}^d$ denote the Φ -split basis for V as in (6), and let $\{v_i\}_{i=0}^d$ denote the Φ -standard basis for V as in (15). Let $T \in \text{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{v_i\}_{i=0}^d$ to $\{u_i\}_{i=0}^d$. Then T is upper triangular with entries

$$T_{ij} = (\theta_i^* - \theta_{i+1}^*) \cdots (\theta_i^* - \theta_{d-1}^*)(\theta_i^* - \theta_d^*) \quad (0 \leq i \leq j \leq d). \quad (16)$$

Moreover T^{-1} is upper triangular with entries

$$T_{ij}^{-1} = \frac{1}{(\theta_j^* - \theta_i^*)(\theta_j^* - \theta_{i+1}^*) \cdots (\theta_j^* - \theta_{j-1}^*)} \frac{1}{(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_{d-1}^*)(\theta_j^* - \theta_d^*)}$$

for $0 \leq i \leq j \leq d$.

Proof. Observe that $u_d = \eta_0 = \sum_{i=0}^d E_i \eta_0 = \sum_{i=0}^d v_i$ and so $T_{id} = 1$ for $0 \leq i \leq d$. Moreover by Lemma 8.6(ii) we have $D^*T = TB^*$, where D^* is from Lemma 8.5 and B^* is from Proposition 4.4. Therefore by a routine matrix multiplication, we find $\theta_i^* T_{ij} = T_{i,j-1} + \theta_j^* T_{ij}$ for $0 \leq i \leq d$ and $1 \leq j \leq d$. Rearranging we obtain $T_{i,j-1} = (\theta_i^* - \theta_j^*) T_{ij}$ and (16) follows by a simple recursion. To prove our assertion about T^{-1} , let $S \in \text{Mat}_{d+1}(\mathbb{K})$ denote an upper triangular matrix with entries

$$S_{ij} = \frac{1}{(\theta_j^* - \theta_i^*)(\theta_j^* - \theta_{i+1}^*) \cdots (\theta_j^* - \theta_{j-1}^*)} \frac{1}{(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_{d-1}^*)(\theta_j^* - \theta_d^*)} \quad (17)$$

for $0 \leq i \leq j \leq d$. It suffices to show that $TS = I$. The matrices T and S are both upper triangular, so TS is upper triangular. By (16), (17) we find that for $0 \leq i \leq d$,

$$S_{ii} = \frac{1}{(\theta_i^* - \theta_{i+1}^*) \cdots (\theta_i^* - \theta_d^*)} = T_{ii}^{-1}$$

so $(TS)_{ii} = 1$. We now show that $(TS)_{ij} = 0$ for $0 \leq i < j \leq d$. Let i, j be given. It suffices to show that $(\theta_i^* - \theta_j^*)(TS)_{ij} = 0$, since $\{\theta_h^*\}_{h=0}^d$ are mutually distinct. Observe that

$$(\theta_i^* - \theta_j^*)(TS)_{ij} = (\theta_i^* - \theta_j^*) \sum_{h=0}^d T_{ih} S_{hj}$$

$$\begin{aligned}
&= (\theta_i^* - \theta_j^*) \sum_{h=i}^j T_{ih} S_{hj} \\
&= \sum_{h=i}^j T_{ih} S_{hj} (\theta_i^* - \theta_h^* + \theta_h^* - \theta_j^*) \\
&= \sum_{h=i+1}^j T_{ih} (\theta_i^* - \theta_h^*) S_{hj} - \sum_{h=i}^{j-1} T_{ih} S_{hj} (\theta_j^* - \theta_h^*) \\
&= \sum_{h=i+1}^j T_{i,h-1} S_{hj} - \sum_{h=i}^{j-1} T_{ih} S_{h+1,j} \\
&= 0
\end{aligned} \tag{18}$$

since the two sums in (18) are one and the same. We have now shown that $(TS)_{ij} = 0$ for $0 \leq i < j \leq d$. Combining our above arguments, we find $TS = I$. The result follows. \square

Proposition 8.8. With reference to Notation 7.1 and Definition 8.4, the matrix H is Hessenberg with entries

$$\begin{aligned}
H_{ij} = & \sum_{h=i}^j \frac{(\theta_i^* - \theta_{h+1}^*) \cdots (\theta_i^* - \theta_{d-1}^*)(\theta_i^* - \theta_d^*) \theta_{d-h}}{(\theta_j^* - \theta_h^*)(\theta_j^* - \theta_{h+1}^*) \cdots (\theta_j^* - \theta_{j-1}^*)(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_{d-1}^*)(\theta_j^* - \theta_d^*)} \\
& + \sum_{h=i-i'}^{j-j'} \frac{(\theta_i^* - \theta_{h+2}^*) \cdots (\theta_i^* - \theta_{d-1}^*)(\theta_i^* - \theta_d^*) \phi_{h+1}}{(\theta_j^* - \theta_h^*)(\theta_j^* - \theta_{h+1}^*) \cdots (\theta_j^* - \theta_{j-1}^*)(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_{d-1}^*)(\theta_j^* - \theta_d^*)},
\end{aligned}$$

for $i \leq j+1$ ($0 \leq i, j \leq d$), where

$$i' = \begin{cases} 0, & \text{if } i = 0 \\ 1, & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad j' = \begin{cases} 0, & \text{if } j \neq d \\ 1, & \text{if } j = d. \end{cases}$$

Proof. By Lemma 8.6(i), $H = TBT^{-1}$, where B is the matrix on the left in (7) and T, T^{-1} are from Lemma 8.7. By a routine matrix multiplication, we find that each entry below the subdiagonal of H is zero and the remaining entries of H are as claimed. To see that H is Hessenberg, it remains to show that each entry on the subdiagonal is nonzero. Using the above data, we find

$$H_{j+1,j} = \frac{(\theta_{j+1}^* - \theta_{j+2}^*) \cdots (\theta_{j+1}^* - \theta_d^*) \phi_{j+1}}{(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_d^*)},$$

for $0 \leq j \leq d-1$. Since $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and $\{\phi_i\}_{i=1}^d$ are nonzero, $H_{j+1,j}$ is nonzero. Therefore H is Hessenberg and the result follows. \square

We give three characterizations of the Φ -standard basis.

Proposition 8.9. With reference to Notation 7.1, let $\{v_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then this sequence is a Φ -standard basis for V if and only if the following (i) and (ii) hold.

- (i) $v_i \in E_i^* V$ for $0 \leq i \leq d$.
- (ii) $\sum_{i=0}^d v_i \in E_0 V$.

Proof. To prove the proposition in one direction, assume that $\{v_i\}_{i=0}^d$ is a Φ -standard basis for V . By Definition 8.2, there exists $0 \neq \eta_0 \in E_0 V$ such that $v_i = E_i^* \eta_0$ for $0 \leq i \leq d$. Apparently $v_i \in E_i^* V$ so (i) holds. Observe that $I = \sum_{i=0}^d E_i^*$. Applying both sides to η_0 we find $\eta_0 = \sum_{i=0}^d v_i$ and (ii) follows. We

have now proved the proposition in one direction. To prove the other direction, assume that $\{v_i\}_{i=0}^d$ satisfy (i) and (ii) above. We define $\eta_0 = \sum_{i=0}^d v_i$ and observe that $\eta_0 \in E_0 V$ by (ii). Using (i) we find $E_i^* v_j = \delta_{ij} v_j$ for $0 \leq i, j \leq d$ and hence $v_i = E_i^* \eta_0$ for $0 \leq i \leq d$. Observe that $\eta_0 \neq 0$ since at least one of $\{v_i\}_{i=0}^d$ is nonzero. Now $\{v_i\}_{i=0}^d$ is a Φ -standard basis for V by Definition 8.2. \square

We recall some notation. For $X \in \text{Mat}_{d+1}(\mathbb{K})$ and $\alpha \in \mathbb{K}$, X is said to have constant row sum α whenever $\sum_{h=0}^d X_{ih} = \alpha$ for $0 \leq i \leq d$.

Proposition 8.10. *With reference to Notation 7.1, let $\{v_i\}_{i=0}^d$ denote a basis for V . Let C (resp. C^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to this basis. Then $\{v_i\}_{i=0}^d$ is a Φ -standard basis for V if and only if the following (i) and (ii) hold.*

- (i) C has constant row sum θ_0 .
- (ii) $C^* = \text{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$.

Proof. Observe that $A \sum_{j=0}^d v_j = \sum_{i=0}^d v_i (C_{i0} + C_{i1} + \dots + C_{id})$. Recall $E_0 V$ is the eigenspace of A corresponding to eigenvalue θ_0 . Apparently C has constant row sum θ_0 if and only if $\sum_{i=0}^d v_i \in E_0 V$. Recall that for $0 \leq i \leq d$, $E_i^* V$ is the eigenspace of A^* corresponding to eigenvalue θ_i^* . Apparently $C^* = \text{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$ if and only if $v_i \in E_i^* V$ for $0 \leq i \leq d$. The result follows by Proposition 8.9. \square

Proposition 8.11. *With reference to Notation 7.1, let $\{v_i\}_{i=0}^d$ denote a basis for V , and let C (resp. C^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to this basis. Then $\{v_i\}_{i=0}^d$ is a Φ -standard basis for V if and only if the following (i) and (ii) hold.*

- (i) C is Hessenberg with constant row sum θ_0 .
- (ii) C^* is diagonal and $C_{00}^* = \theta_0^*$.

Proof. Suppose that $\{v_i\}_{i=0}^d$ is a Φ -standard basis for V . By Proposition 8.8, C is Hessenberg and by Proposition 8.10(i), C has constant row sum θ_0 . By Proposition 8.10(ii), C^* is diagonal and $C_{00}^* = \theta_0^*$. Therefore (i) and (ii) above hold. We have proved the proposition in one direction. To prove the other direction, suppose that (i) and (ii) above hold. Since C is Hessenberg and C^* is diagonal, applying Lemma 3.4 we find $\{C_{ii}^*\}_{i=0}^d$ is a dual eigenvalue sequence of A , A^* . Since $C_{00}^* = \theta_0^*$, applying Lemma 3.3 we find $C_{ii}^* = \theta_i^*$ for $0 \leq i \leq d$. Therefore $C^* = \text{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$ and by Proposition 8.10, $\{v_i\}_{i=0}^d$ is a Φ -standard basis for V . \square

9. The Φ^* -standard basis

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V . In the previous section, we obtained several results related to the Φ -standard basis for V . Analogous results hold for the Φ^* -standard basis for V . In this section, we display some of those results for use later in the paper. We begin by observing that a Φ^* -standard basis for V has the form

$$E_0 \eta_0^*, E_1 \eta_0^*, \dots, E_d \eta_0^*, \quad (19)$$

where $0 \neq \eta_0^* \in E_0^* V$.

Definition 9.1. With reference to Notation 7.1, let D (resp. H^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to a Φ^* -standard basis for V .

Lemma 9.2. *With reference to Notation 7.1 and Definition 9.1, the matrix D is diagonal with entries $D_{ii} = \theta_i$ for $0 \leq i \leq d$.*

Proof. Apply Lemma 8.5 to Φ^* . \square

Lemma 9.3. With reference to Notation 7.1 and Definition 9.1, let $\{u_i\}_{i=0}^d$ denote a Φ^* -split basis for V , and let $\{v_i\}_{i=0}^d$ denote a Φ^* -standard basis for V . Let B (resp. B^*) denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A (resp. A^*) with respect to $\{u_i\}_{i=0}^d$. Let $T^* \in \text{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{v_i\}_{i=0}^d$ to $\{u_i\}_{i=0}^d$. Then the following (i) and (ii) hold.

- (i) $D = T^* B T^{*-1}$.
- (ii) $H^* = T^* B^* T^{*-1}$.

Proof. Apply Lemma 8.6 to Φ^* . \square

Lemma 9.4. With reference to Notation 7.1, fix $0 \neq \eta_0^* \in E_0^* V$. Let $\{u_i\}_{i=0}^d$ denote the Φ^* -split basis for V as in (10), and let $\{v_i\}_{i=0}^d$ denote the Φ^* -standard basis for V as in (19). Let $T^* \in \text{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{v_i\}_{i=0}^d$ to $\{u_i\}_{i=0}^d$. Then T^* is upper triangular with entries

$$T_{ij}^* = (\theta_i - \theta_{j+1}) \cdots (\theta_i - \theta_{d-1})(\theta_i - \theta_d) \quad (0 \leq i \leq j \leq d).$$

Moreover T^{*-1} is upper triangular with entries

$$T_{ij}^{*-1} = \frac{1}{(\theta_j - \theta_i)(\theta_j - \theta_{i+1}) \cdots (\theta_j - \theta_{j-1})} \frac{1}{(\theta_j - \theta_{j+1}) \cdots (\theta_j - \theta_{d-1})(\theta_j - \theta_d)}$$

for $0 \leq i \leq j \leq d$.

Proof. Apply Lemma 8.7 to Φ^* . \square

Proposition 9.5. With reference to Notation 7.1 and Definition 9.1, the matrix H^* is Hessenberg with entries

$$H_{ij}^* = \sum_{h=i}^j \frac{(\theta_i - \theta_{h+1}) \cdots (\theta_i - \theta_{d-1})(\theta_i - \theta_d) \theta_{d-h}^*}{(\theta_j - \theta_h)(\theta_j - \theta_{h+1}) \cdots (\theta_j - \theta_{j-1})(\theta_j - \theta_{j+1}) \cdots (\theta_j - \theta_{d-1})(\theta_j - \theta_d)} \\ + \sum_{h=i-i'}^{j-j^\vee} \frac{(\theta_i - \theta_{h+2}) \cdots (\theta_i - \theta_{d-1})(\theta_i - \theta_d) \phi_{d-h}}{(\theta_j - \theta_h)(\theta_j - \theta_{h+1}) \cdots (\theta_j - \theta_{j-1})(\theta_j - \theta_{j+1}) \cdots (\theta_j - \theta_{d-1})(\theta_j - \theta_d)},$$

for $i \leq j+1$ ($0 \leq i, j \leq d$), where

$$i' = \begin{cases} 0, & \text{if } i = 0 \\ 1, & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad j^\vee = \begin{cases} 0, & \text{if } j \neq d \\ 1, & \text{if } j = d. \end{cases}$$

Proof. Apply Proposition 8.8 to Φ^* . \square

10. Transition matrices between the Φ -standard basis and the Φ^* -standard basis

In this section we describe the transition matrices between the Φ -standard basis for V and the Φ^* -standard basis for V .

Definition 10.1. With reference to Notation 7.1, let Z denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ with entries

$$Z_{ij} = \delta_{i,d-j} \frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)}{\phi_{i+1} \phi_{i+2} \cdots \phi_d}$$

for $0 \leq i, j \leq d$.

The following lemma gives the significance of Z .

Lemma 10.2. With reference to Notation 7.1, let $\{v_i\}_{i=0}^d$ denote a Φ -split basis for V as in (6), and let $\{w_i\}_{i=0}^d$ denote a Φ^* -split basis for V as in (10) with η_0, η_0^* from those lines chosen so that $\eta_0 = E_0\eta_0^*$. Then the transition matrix from $\{v_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$ is the matrix Z from Definition 10.1.

Proof. The transition matrix from $\{v_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$ is given by Lemma 5.4 for an appropriate value of the scalar c in that lemma. We now find that value. Setting $i = 0$ in Lemma 5.4(ii), we find $w_0 = cv_d$. Using (10), we find $w_0 = (A - \theta_1 I)(A - \theta_2 I) \cdots (A - \theta_d I)\eta_0^*$ and this equals $(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)E_0\eta_0^*$. Using (6), we find $v_d = \eta_0 = E_0\eta_0^*$ by our choice of η_0, η_0^* . Therefore $w_0 = (\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)v_d$ and so $c = (\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)$. The result follows. \square

Definition 10.3. With reference to Notation 7.1, let Z^* denote the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ with entries

$$Z_{ij}^* = \delta_{i,d-j} \frac{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{\phi_1 \phi_2 \cdots \phi_{d-i}}$$

for $0 \leq i, j \leq d$.

The following lemma gives the significance of Z^* .

Lemma 10.4. With reference to Notation 7.1, let $\{v_i\}_{i=0}^d$ denote a Φ -split basis for V as in (6), and let $\{w_i\}_{i=0}^d$ denote a Φ^* -split basis for V as in (10) with η_0, η_0^* from those lines chosen so that $\eta_0^* = E_0^*\eta_0$. Then the transition matrix from $\{w_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is the matrix Z^* from Definition 10.3.

Proof. Apply Lemma 10.2 to Φ^* . \square

Lemma 10.5. With reference to Notation 7.1 and Definitions 10.1, 10.3,

$$ZZ^* = vI, \quad Z^*Z = vI,$$

where v is the scalar above Lemma 7.4.

Proof. Multiply Z and Z^* and use Lemma 7.6. \square

Definition 10.6. With reference to Notation 7.1, let P (resp. P^*) denote the transition matrix from (19) to (15) (resp. (15) to (19)) with η_0, η_0^* from those lines chosen so that $\eta_0^* = E_0^*\eta_0$ (resp. $\eta_0 = E_0\eta_0^*$).

Lemma 10.7. With reference to Notation 7.1 and Definition 10.6,

$$P = T^*Z^*T^{-1}, \quad P^* = TZT^{*-1},$$

where T, T^* are from Lemmas 8.7, 9.4 and Z, Z^* are from Definitions 10.1 and 10.3.

Proof. By Lemma 8.7, T^{-1} is the transition matrix from (6) to (15). Moreover by Lemma 10.4, Z^* is the transition matrix from (10) to (6) and by Lemma 9.4, T^* is the transition matrix from (19) to (10). Therefore, by the comments prior to Lemma 8.6, we find $T^*Z^*T^{-1}$ is the transition matrix from (19) to (15) and hence equals P . We have proved the equation on the left. To prove the equation on the right, apply the equation on the left to Φ^* . \square

Theorem 10.8. With reference to Notation 7.1 and Definition 10.6, the following (i) and (ii) hold.

(i) For $0 \leq i, j \leq d$, P_{ij} is equal to

$$\frac{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{(\theta_j^* - \theta_0^*)(\theta_j^* - \theta_1^*) \cdots (\theta_j^* - \theta_{j-1}^*)(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_{d-1}^*)(\theta_j^* - \theta_d^*)}$$

times

$$\sum_{h=0}^d \frac{(\theta_i - \theta_d)(\theta_i - \theta_{d-1}) \cdots (\theta_i - \theta_{d-h+1})(\theta_j^* - \theta_0^*)(\theta_j^* - \theta_1^*) \cdots (\theta_j^* - \theta_{h-1}^*)}{\phi_1 \phi_2 \cdots \phi_h}.$$

In particular $P_{i0} = 1$ for $0 \leq i \leq d$.

(ii) For $0 \leq i, j \leq d$, P_{ij}^* is equal to

$$\frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)}{(\theta_j - \theta_0)(\theta_j - \theta_1) \cdots (\theta_j - \theta_{j-1})(\theta_j - \theta_{j+1}) \cdots (\theta_j - \theta_{d-1})(\theta_j - \theta_d)}$$

times

$$\sum_{h=0}^d \frac{(\theta_i^* - \theta_d^*)(\theta_i^* - \theta_{d-1}^*) \cdots (\theta_i^* - \theta_{d-h+1}^*)(\theta_j - \theta_0)(\theta_j - \theta_1) \cdots (\theta_j - \theta_{h-1})}{\phi_d \phi_{d-1} \cdots \phi_{d-h+1}}.$$

In particular $P_{i0}^* = 1$ for $0 \leq i \leq d$.

Proof. Routine by Lemma 10.7 and matrix multiplication. \square

Proposition 10.9. With reference to Notation 7.1 and Definition 10.6,

$$PP^* = \nu I, \quad P^*P = \nu I,$$

where ν is the scalar above Lemma 7.4.

Proof. Evaluate the left hand sides using Lemma 10.7 and then simplify using Lemma 10.5. \square

We finish this section with a comment.

Lemma 10.10. With reference to Notation 7.1 and Definition 10.6,

$$H = P^{-1}DP, \quad H^* = P^{*-1}D^*P^*,$$

where D^*, H (resp. D, H^*) are from Definition 8.4 (resp. Definition 9.1).

Proof. Recall from Definition 8.4 (resp. Definition 9.1) that H (resp. D) is the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ which represents A with respect to a Φ -standard (resp. Φ^* -standard) basis for V . Moreover recall from Definition 10.6 that P is a transition matrix from a Φ^* -standard basis to a Φ -standard basis. Therefore the equation on the left holds by the comments prior to Lemma 8.6. To prove the equation on the right, apply the equation on the left to Φ^* . \square

Acknowledgments

This paper was written while the author was a graduate student at the University of Wisconsin-Madison. The author would like to thank his advisor Paul Terwilliger for his many valuable ideas and suggestions.

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